Paper 1 Economics Supervision 4: Welfare Economics (Welfare Theorems)

Biagio Rosso [1]

[1] Centre of Development Studies, University of Cambridge
br421@cam.ac.uk

Jan 30-31, 2025

Outline

Elements of Welfare Economics

- Assessing the efficiency of an economy / allocation mechanism relative to some pre-defined objective or criterion.
- What criterion?
- In this microeconomics/price theory course, we are concerned with the efficiency of (static) *decentralised or competitive market equilibria*.
- Why?
  1. Whether and under what conditions an existing market / price system arrangements for allocating a set of commodities achieve a Pareto Optimal allocation (WT1).
  2. Whether and how some centralised socially optimal allocation (a benevolent planner's allocation) can be supported through an unregulated market mechanism/price system (rather than, say central planning/price regulations, etc). Institutional design.
  3. There are some practical applications to WT2 other than welfare econ (solving for equilibrium more easily), but require strict assumptions.

Cont's

- Discussing this topic will allow us to distinguish between an allocative function of prices (and its failure), and a distributive function of prices.
- These closely related and best understood in relation to WT1 and WT2.
- Idea in assignment is drawing on these for Part A, and similarly in your analysis for Part B.
- Necessary technical background:
  1. Competitive Equilibrium (Consumer Problem + Feasibility/Market Clearing). Critical assumptions.
  2. Optimisation theory: solution to Lagrangian/KKT Conditions necessary and sufficient for optimum.
  3. Pareto Efficiency.
  4. Planner's Problem.
- I'll formally demonstrate and explain the key idea on which the WTs rest + use this logic to discuss market failure with externalities.

Competitive Equilibrium / Walras Equilibrium

- Consider an economy $\mathcal{E} = \{u_i, \mathbf{c}_i, \mathbf{e}_i\}_{i \in \mathbb{N}}$.
- Many price taking agents, perfect information...
- Key further Assumptions on $u_i : \mathbb{R}^n \to \mathbb{R}$
- Twice differentiable in all arguments
- Preferences are monotonic (strictly increasing in each argument): $u'_{c_j}(\mathbf{c}) > 0, \quad \forall j, \forall \mathbf{c} \in \mathbb{R}^n$. Hence the Jacobian $Du(\mathbf{c}) > 0 \forall \mathbf{c} \in \mathbb{R}^n$.
- Preferences are strictly convex (=*strictly quasi-concave* utility function u_i). This implies that, for any two commodity bundles/vectors:
  1. $$\mathbf{u}_i(\mathbf{c}') \geq u_i(\mathbf{c}) \to u_i(\alpha \mathbf{c} + (1-\alpha)\mathbf{c}) > u_i(\mathbf{c})$$
  2. $$Du(\mathbf{c})(\mathbf{c}' - \mathbf{c}) > 0$$

     whenever $\mathbf{c}' \geq \mathbf{c}$ (element-wise)
- No externalities (requires, e.g. utility only depends on own consumption bundle)
- Convex budget constraints/budget sets.

General Competitive Equilibrium / Walras Equilibrium

- A competitive equilibrium/Walras Equilibrium is an pair $\mathcal{S} = (\{\tilde{\mathbf{c}}_{\mathbf{i}}\}_i, \mathbf{p})$ consisting of a bundle for each consumer (an allocation) and a price vector $\mathbf{p} \in \mathbb{R}^n$, such that:
- *Optimal Behaviour*: All agents/households are at their optimal bundle at $\mathbf{c}_i$ when facing (common) price vector $\mathbf{p}$ and their budget constraints. By constrained optimisation theory, we know from week 1 this is the case *iff*

$$u'_{c_j}(\tilde{\mathbf{c}}) = \lambda p_j$$

$$\mathbf{p}'\tilde{\mathbf{c}}_i = w$$

- First can be rewritten stacking across columns as

$$Du(\tilde{\mathbf{c}}_{\mathbf{i}}) = \lambda_i \mathbf{p}'$$

- Questions on where this comes from?
- Lagrangian multiplier?

General Competitive Equilibrium / Walras Equilibrium

- In other words: each agent i MUST HAVE NO incentive to change from (candiate equilibrium) bundle $\tilde{\mathbf{c}}_{\mathbf{i}}$ to another feasible bundle when facing (candidate equilibrium) prices $\mathbf{p}$. I.e. Tangency Condition + Budget Constraint.
- If such an incentive exists, then we can't be at an optimum. Can you show this graphically?
- The allocation must be... an allocation = it must be feasible given resource constraints. Sum of individual consumptions = total endowment in that good.

Pareto Efficiency

- An allocation $\{\tilde{\mathbf{c}}_i\}_i$ is PARETO EFFICIENT if there is NO other allocation $\{\bar{\mathbf{c}}_\mathbf{i}\}_i$ such that

$$u_i(\bar{\mathbf{c}}_\mathbf{i}) \geq u_i(\tilde{\mathbf{c}}_\mathbf{i})$$

  for all agents i, with STRICT equality for some.
- No other allocation such that everyone is at least as better off, and there is at least some agent who is strictly better off.

Pareto Efficiency

- Is the decentralised equilibrium/competitive equilibrium/price mechanism described earlier a Pareto Efficient allocation mechanism?
- In other words, can we find an allocation (e.g. centralised, random, etc) that pareto-dominates the one achieved by the price-mechanism?
- **First Welfare Theorem (WT1)** says that, under the listed conditions/assumptions, we cannot.
- The efficiency of markets/WT1 (a positive notion) undergirds the normative take-away that one should not tamper with the price-mechanism. **Policy**: Get prices right/leave the invisible hand to work its way/Laissez Faire.
- The idea of the *allocative* function of prices is at the heart of the WT1 and policy take-aways. See later.

Proof of WT1

- Assume, toward a contradiction, that the market allocation is not Pareto Efficient.
- By definition, there must be at least another allocation $\{\bar{\mathbf{c}}_\mathbf{i}\}_i$ such that

$$u_i(\bar{\mathbf{c}}_\mathbf{i}) \geq u_i(\tilde{\mathbf{c}}_\mathbf{i})$$

  for all agents i, with STRICT equality for some.
- Without loss of generality, suppose that such allocation is of the form

$$\bar{\mathbf{c}}_\mathbf{i} = \tilde{\mathbf{c}}_\mathbf{i}, \qquad \text{for agents} \quad i \neq k$$

$$\bar{\mathbf{c}}_\mathbf{k} \geq \tilde{\mathbf{c}}_\mathbf{k}, \qquad \text{for agent} \quad k$$

- But then, by MONOTONICITY

$$u_k(\bar{\mathbf{c}}_\mathbf{k}) > u_k(\tilde{\mathbf{c}}_\mathbf{k})$$

  with strict equality for some commodity in the bundle $\mathbf{c}_k$
- Intuition is: we are constructing an example in which at least an agent k would be better off. Hence, if the allocation is feasible, they would have an incentive to deviate.

Proof of WT1

- Assume, toward a contradiction, that the market allocation is not Pareto Efficient.
- There must be at least another allocation $\{\bar{\mathbf{c}}_\mathbf{i}\}_i$ such that

$$u_i(\bar{\mathbf{c}}_\mathbf{i}) \geq u_i(\tilde{\mathbf{c}}_\mathbf{i})$$

  for all agents i, with strict equality for some.
- Without loss of generality, suppose that such allocation is of the form

$$\bar{\mathbf{c}}_\mathbf{i} = \tilde{\mathbf{c}}_\mathbf{i}, \qquad \text{for agents} \quad i \neq k$$

$$\bar{\mathbf{c}}_\mathbf{k} \geq \tilde{\mathbf{c}}_\mathbf{k}, \qquad \text{for agent} \quad k$$

- But then, by monotonicity of preferences

$$u_k(\bar{\mathbf{c}}_\mathbf{k}) > u_k(\tilde{\mathbf{c}}_\mathbf{k})$$

  with strict equality for some commodity in the bundle $\mathbf{c}_k$
- Intuition is: we are constructing an example in which at least an agent k would be better off. Hence, if the allocation is feasible at current prices, they would have an incentive to deviate.

Cont'd

- The proof by contradiction shows that, if the original allocation is a competitive equilibrium, then such a feasible allocation cannot exist.
- 1: By the definition of competitive equilibrium, we are at a consumer optimum/tangency conditions holds and budgets holds tightly. That is

$$Du(\tilde{\mathbf{c}}_k) = \lambda_k \mathbf{p}'$$

$$\mathbf{p}'\tilde{\mathbf{c}}_k = w_k$$

- But then, by strict-quasi concavity / convexity of preferences $u_k(\bar{\mathbf{c}}_\mathbf{k}) > u_k(\tilde{\mathbf{c}}_\mathbf{k})$ implies: $Du(\tilde{\mathbf{c}}_\mathbf{k})(\bar{\mathbf{c}}_\mathbf{k} - \tilde{\mathbf{c}}_\mathbf{k}) > 0$

Cont'd

- But then, using the tangency/optimality conditions and tightness of budget constraint:

$$\lambda_k \mathbf{p}' \bar{\mathbf{c}}_{\mathbf{k}} > \lambda_k \mathbf{p}' \tilde{\mathbf{c}}_{\mathbf{k}}$$

- We know by lagrangian optimisation that, if the budget constraint binds, then the lagrangian multiplier or price of violations of the budget constraint $\lambda_k > 0$.

$$\mathbf{p}' \bar{\mathbf{c}}_{\mathbf{k}} > w_k$$

- Therefore, the bundle $\bar{\mathbf{c}}_{\mathbf{k}}$ and hence the hypothesised Pareto-Dominant allocation $\{\bar{\mathbf{c}}_{\mathbf{i}}\}_i$ is not feasible.
- Informally... + Edgeworth Box.

WT1 Interpretation and the Allocative Function of Prices

- ...the price-mechanism, under the listed assumptions, delivers an allocation from which individuals have no incentive to deviate and such that it exhausts their budget constraints. In particular, with the budget constraint holding tightly (which follows from LNS), at the proposed bundle the rate of increase in utility in the last unit just offsets the penalty they would incur for violating their budget constraints.
- Then, a Pareto-Dominant or Pareto-Superior allocation relative to the market one – such to generate higher utility for at least one agent and leaving all the other ones indifferent, cannot be feasible, as it would involve a violation of the budget constraint at current prices.
- A Central Planner cannot do better than the market. All we can do is redistribute so that another Pareto-Optimal allocation prevails through the market mechanism (...see WT2)

Cont'd

- Closely tied to a notion of allocative function of prices. The price-mechanism/decentralised equilibrium is efficient because (a) prices signal the market valuation of a good/service (ex post), and (b) the resulting allocation is such to equalise the marginal utility of agents across goods/services, AND (c) hence equate it for each agent to the market valuation.
- This rests, critically, as we have seen, on local non-satiation of preferences, which results in strictly positive penalties (the lagrange/KKT multipliers) attached to the violation of constraints (budgets) on individual optimisation.
- When this is the case, given a common price vector, agents who value more an extra unit of the good end up receiving a larger amount of it (as in earlier slide).
- How can we link this optimisation/mathematical treatment to the efficiency of the allocation (informally?)

Cont'd

- Some Ideas
- (1) Prices signal the "correct" marginal allocations/rate for substitutions between different goods, which will be the same across consumers under a market/price-mechanism allocation.
- (2) The price mechanism allocates more of a resource to those who value it (marginally/an extra unit of it) more as long as they can afford it.
- (3) Because there are no externalities, then by construction the marginal valuation/private marginal utility of agent i from consuming some amount of a commodity is the same as that of society.
- (4) Hence, in some sense, the efficiency/social optimality of the price mechanism allocation comes from the fact that it assigns larger consumptions/bundles to those agents whose consumption society values more highly...

Limits to WT1

- Failure of assumption on preferences (e.g. LNS)
- Transaction costs.
- More typical: Externalities and Incomplete Markets.
- Second, no idea about fairness/equity of allocations. Need to wait for WT2.

Externalities

- Case of consumption externalities. Two "representative" agents (many within each type... price taking): agents whose consumption has externalities and those who suffer from it. Formally, utility of latter depends on consumption of former. Since the latter cannot choose or buy/sell consumption of former, or write contracts, markets are incomplete.
- Many goods, but only one assumed to have an externalities, we let this good c_e. Partition bundle...
- Agent 1: $u(\mathbf{c}^1_{-e}, c^1_e)$
- Agent 2: $v(\mathbf{c}^2_{-e}, c^2_e, c^1_e)$
- Negative externality (use of pollutants from upstream landplot) or positive externalities possible.

Externalities

- Market/competitive equilibrium satisfies, inter alia:
  $u'_{c_e^1}(\mathbf{c}) = \lambda_1 p_e$
- SWF approach. Social optimal allocation – including of c^{1*} – (placing equal weights on two agents), under the same constraint/price vector, would require:
  $u'_{c_e^1}(\mathbf{c}^{1*}) + v'_{c_e^1}(\mathbf{c}^{2*}) = \lambda_1^* p_e$
- $v'_{c_e^1} > 0$ is a positive externality and viceversa.
- The decentralised optimum coincides with the optimal allocation (at such prices) only if there are no externalities.
- Just an intuition (need to construct planner problem explicitly). With negative (positive) externalities MSB will lie below (above) the PSB at any quantity allocated.

The Second Welfare Theorem and Distributive Role of Prices

- The WT2 is an answer (I believe limitative) to two questions
- 1. How can we implement a socially optimal allocation? Answer: *under the listed conditions, a socially optimal allocation can always be achieved as a competitive equilibrium, i.e. there is some price vector that decentralised the allocation...*
- 2. What policy tools? *...provided lump-sum transfers are available to the planner to redistribute initial endowments*.
- As I have come to see it (partly as a macroeconomist), two standard interpretations.

Cont'd

- **View I – Neo-classical Welfare Economics**
- First, that the planner/government should not tamper with prices when attempting to achieve some desirable or different allocation. Rather, focus should be on the non-distortionary (lump-sum) redistribution of endowments.
- This highlights/builds on a *distributive function* of prices in market mechanisms, and say it's separate from the allocative one. Distributive function = prices determine budget set and cash-on-hand = how much of different goods individuals can buy.
- *WT2 tells us that the two functions can be separated through the use of appropriate policy tools: redistributive lump-sum transfers on endowments and leaving prices alone.*
- Because this allows to separate the two functions, the allocative/signalling function of prices responsible for efficiency (as in WT1) continues to apply, so that margins are not distorted, while we exploit their distributive function to implement different initial conditions.

Cont'd

- **View II – Macroeconomists and Fiscal-Monetary Theorists :-)**
- This is a limitative result, as in general the two functions cannot be separated, due to a lack of the required policy tools. Hence the WT2 is a nice result in theoretical/mathematical econ, but with little value to actual policy design other than as an ideal benchmark.
- Inability to redistribute endowments through individualised and lump-sum transfers is key. In general practically available instruments tends to fall short of the required benchmark and are distortionary in nature (e.g. linear taxes on labour and capital income), and sometimes even restricted in the ability to update them over time and the information they can respond to.
- Huge repercussions for how we solve macro models and discuss optimal policy (Ramsey Programming, "Simple Rules" literature)
- Hence, nice to know, but practically irrelevant.

Proof of WT2

- I am going to prove a version of the theorem to get the main idea.
- We show that the allocation that solves the planner problem constitutes a Walras Equlibrium for an economy with a price vector corresponding to the vector of lagrange multiplier on the aggregate resource constraints. I.e. it can be supported by a vector of prices matching a measure of scarcity of the goods at a social level.
- Key to implementing this is, again, the idea that the government can redistribute endowments in a non-distortionary way.
- I am gonna construct for you a state-of-the-art planner problem.

Planner Problem

- Economy again with I agents and K commodities.
- Suppose we are at some allocation $\{\tilde{\mathbf{c}}_\mathbf{i}\}_i$. The planner problem is to choose (and implement) another allocation solving, given some arbitrary agent 1:

$$\max_{\{\mathbf{c}_j\}_{j\in I}} u_1(\mathbf{c}_1)$$

$$s.t.$$

$$u_i(\mathbf{c}_i) \geq u_i(\tilde{\mathbf{c}}_\mathbf{i}), \qquad \forall i \in I/1 \qquad (ICs)$$

$$\sum_i c_{i,k} \geq \sum_i e_{i,k} \qquad (Aggregate\ Res.\ Constraints)$$

- Note no prices... centralised allocation (hence can disregard these).

Solution

- Constrained optimisation with inequalities (although we'll see they always hold tight – hence Lagrane theorem suffices...)
- By the KKT Theorem, under monotonicity and concavity of preferences etc, an allocation can be found by maximisation of the Lagrangian

$$\max_{\{\mathbf{c}_j\}_{j\in I},\{\alpha_i\}_{i\in I},\{\lambda_k\}_{k\in K}} u_1(\mathbf{c}_1) + \sum_{i\in I} \alpha_i[u_i(\mathbf{c}_i) - u_i(\tilde{\mathbf{c}}_{\mathbf{i}})]$$

$$- \sum_{k\in K} \lambda_k[\sum_{i\in I} c_{i,k} - \sum_{i\in I} e_{i,k}]$$

Cont'd

- Let λ the column vector given by stacking the respective multipliers on the K aggregate resource constraints. The solution is given by the KKT conditions (square system...):

$$Du_1(\mathbf{c}_1) = \lambda$$

$$\alpha_i Du_i(\mathbf{c}_i) = \lambda \quad \forall i \neq 1$$

$$\alpha_i[u_i(\mathbf{c}_i) - u_i(\tilde{\mathbf{c}}_\mathbf{i})] = 0 \quad \forall i \neq 1 \quad (CS)$$

$$\lambda_k[\sum_{i \in I} c_{i,k} - \sum_{i \in I} e_{i,k}] = 0 \quad \forall k \in K \quad (CS)$$

- Suppose first constraint doesn't bind. Then $\alpha_i = 0$ by the CS condition. But then, $\lambda = 0$, which contradicts monotonicity of preferences / LNS for the first agent. Intuition: if monotonic objective, then always optimal to keep the utility of other agents strictly at the current reservation level.

Cont'd

- Let the Agg. Resource constraint not bind for some k. Then by CS, we have $\lambda = 0$, and same issue as earlier applied. I.e. the utility of all agents (hence of agent 1) can be increased by moving from the interior to fully exhausting the Agg. Resource constraint.
- For all k...
- Hence, constraints bind with equality and $\lambda > 0$ $\alpha_k > 0$ for all k.
- Then we have, that at the Social Planner Optimum:

$$Du_1(\mathbf{c}_1) = \lambda$$

$$\alpha_i Du_i(\mathbf{c}_i) = \frac{1}{\alpha_i}\lambda \quad \forall i \neq 1$$

$$\sum_{i \in I} c_{i,k} = \sum_{i \in I} e_{i,k} = 0 \quad \forall k \in K$$

- Denote the solution for the allocation to each agent as $\bar{\mathbf{c}}_\mathbf{i}$

Cont'd

- But this is just the definition of a Walras Equilibrium where the price vector is λ and the agents' lagrange multipliers are $\{1, \{\alpha_i\}_{i \in I}\}$.
- In particular, suppose that *lump-sum individual transfers are available* so that the planner can assign $\mathbf{c}_i$ solving the above problem as the endowment to the agent $\mathbf{e}_i = \mathbf{c}_i$, and then markets open (let agents trade).
- At a decentralised/competitive equilibrium given such endowment vector and some price vector, for all i:

$$Du_i(\mathbf{c}_i^*) = \gamma_i \mathbf{p}'$$

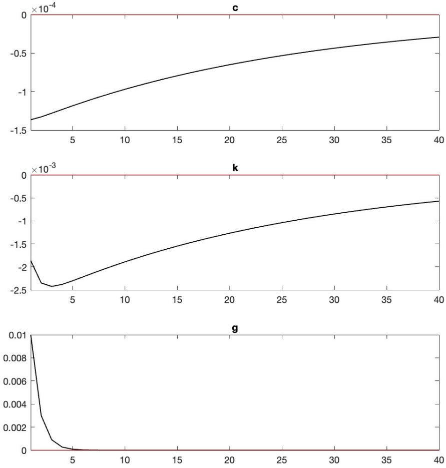
$$\mathbf{p}'\mathbf{c}_i^* = \mathbf{p}'\bar{\mathbf{c}}_\mathbf{i}$$

- Clearly, the planner's optimal allocation $(\bar{c}_i)$ can bealways then supported as part of an equilibrium pair $(\bar{c}_i, \mathbf{p})$, in particular see this by setting $\mathbf{p}' = \lambda$ and $\gamma_i = \frac{1}{\alpha_i}$.
- Edgeworth box

My Analysis/Example: Fiscal Policy Tools

- Raising taxes for financing public expenditure in a flexible-price economy. Aggregates: Consumption, Savings (Capital).
- We might want to decentralise an optimal allocation in the presence of sustainable public expenditure. No debt = expenditure on government consumption must be financed entirely through taxation today.The expenditure process is stochastic
- WT2: If lump-sum redistributive transfers (with unrestricted updating over time) are available as tools, then we know problem is simple.
- But frequently, distortionary tools are the only available: cannot finance expenditure through the above tool. Here, consider taxes on capital income (capital inelastically supplied whenever taxes are revealed).
- How does public expenditure introduce distortionary costs to the economy due to the nature of policy instruments/breakdown of the Second Welfare Theorem?

Case 1 (WT2)


c
×10^-4
0
-0.5
-1
-1.5
5
10
15
20
25
30
35
40
k
×10^-3
0
-0.5
-1
-1.5
-2
-2.5
g
0.01
0.008
0.006
0.004
0.002
0

Case 2 Distortionary Tools